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Pointwise Convergence of Lagrange Interpolation Based at the Zeros of Orthonormal Polynomials with Respect to Weights on the Whole Real Line*

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Under various assumptions on a weight W^2, with support $\mathbb{R}$, we obtain rates for the pointwise convergence of Lagrange interpolation based at the zeros of the orthonormal polynomials with respect to W^2, in the case of a uniformly continuous function $f(x)$. The weights considered include $W_m(x) = \exp(-\frac{1}{2}|x|^m)$, m an even positive integer. The technique used generalizes that of Freud, who considered pointwise convergence of Lagrange interpolation in the case of the Hermite weight. However, even for the Hermite weight, our results refine and extend the upper and lower bounds of Freud. We establish as well, as preliminary results, upper and lower bounds for generalized Lebesgue functions and for absolute values of the orthogonal polynomials associated with $W_m^2(x)$. © 1987 Academic Press, Inc.

1. INTRODUCTION

Convergence of Lagrange interpolation based at the zeros of orthogonal polynomials is a subject which has been widely investigated in the case of weights on a finite interval. For a comprehensive survey of what has been achieved, see Nevai [25, 21]. However, owing to the present dearth of results on orthonormal polynomials on the whole real line, Lagrange interpolation for weights with unbounded support has been investigated primarily in the case of the Hermite weight.

Pointwise convergence of Lagrange interpolation for the Hermite weight was proved first by Freud [6], while Nevai [23] proved results on mean convergence of Lagrange interpolation. Bonan [2] obtained necessary and sufficient conditions for the mean convergence of Lagrange interpolation for the weights $|x|^\alpha \exp(-x^2)$, $\alpha > -1$, in L_p, $0 < p \leqslant \infty$. Sharp results for the pointwise convergence of Lagrange interpolation for weights

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$|x|^\alpha \exp(-x^2)$, $\alpha > -1$, were established by Kis [9]. The case of the Laguerre weight has been investigated in detail by Nevai [18-20]. Recently Knopfmacher and Lubinsky [12] considered mean convergence of Lagrange interpolation for a general class of Freud weights.

In this paper we prove pointwise convergence of Lagrange interpolation for a subclass of the weights $W^2 = \exp(-2Q(x))$, introduced by Freud [8], and which include the weights $W_m(x) = \exp(-\frac{1}{2}|x|^m)$, m an even, positive integer. The technique used, involves a generalization of the ideas of Freud [6]. In addition we make extensive use of properties of exponential weights proved by Freud [8] and bounds for orthonormal polynomials proved by Bonan [3] and Nevai [24].

However, even for the Hermite weight, our results extend those of Freud in one major aspect. The bounds Freud obtained in Satz 2 [6], take no account of the relative position of x and the zeros x_{kn} of the orthonormal polynomial $p_n(x)$ associated with W^2. We show, as one might expect, that since the Lagrange interpolation polynomial interpolates to the function at the zeros x_{kn}, one can obtain enhanced rates of convergence for values of x suitably close to a zero x_{kn}. This is shown in Theorem 3.2, for all the weights considered.

Freud proved, in addition, that for a suitably defined function $f(x)$, his bounds were sharp, but only for certain discrete values of x, namely the zeros of $p_{n+1}(x)$. In Theorem 3.4 we show that our bounds are in general sharp for all values of x lying in an interval which can grow with n.

In Section 2, we introduce the notation that will be used throughout the paper. Furthermore, we state our main results in Section 3, and they are proved in Sections 4 and 5.

2. Notation

Let W denote an even, nonnegative function on $\mathbb{R}$ with all moments

$$\mu_n = \int_{-\infty}^{\infty} x^n W^2(x)\, dx, \qquad n = 0, 1, 2, \ldots, \quad \text{finite.}$$

Also let $\{p_n(W^2, x)\} = \{p_n(x)\}$ be the sequence of orthonormal polynomials with respect to W^2, that is,

$$\int_{-\infty}^{\infty} p_m(x)\, p_n(x)\, W^2(x)\, dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

Let γ_n be the leading coefficient of p_n, $n = 0, 1, 2, \ldots$. Let $a_n = \gamma_{n-1}/\gamma_n$, $n = 1, 2, 3, \ldots$. We assume throughout that $W(x) = \exp(-Q(x))$, where $Q(x)$ is even positive and twice differentiable in $(0, \infty)$ and Q' is positive and nondecreasing in $(0, \infty)$. These weights were considered in some detail by Freud ([7, 8] and references therein). Let q_n denote the unique positive solution of the equation

$$q_n Q'(q_n) = n. \tag{2.1}$$

Note that the sequence $\{q_n\}$ is increasing and as shown in [8, p. 22],

$$1 < q_{2n}/q_n < 2.$$

In keeping with the notation of Freud and others, $K_n(x, y)$ denotes the nth kernel of the orthogonal expansion,

$$\begin{aligned} K_n(x, y) &= \sum_{k=0}^{n-1} p_k(x) p_k(y) \\ &= \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(y) - p_n(y) p_{n-1}(x)}{x - y}, \end{aligned} \tag{2.2}$$

(the Christoffel–Darboux formula) and $\lambda_n(W^2, x) \equiv \lambda_n(x)$ denotes the Christoffel function

$$\lambda_n(W^2, x) = 1/K_n(x, x).$$

Furthermore

$$\lambda_{kn} = \lambda_n(W^2, x_{kn}), \qquad k = 1, 2, ..., n.$$

We denote the zeros of $p_n(x)$ by

$$x_{kn}, \qquad k = 1, 2, ..., n, \text{ where } x_{nn} < x_{n-1,n} < \cdots < x_{1n}.$$

Throughout given x, let x_{jn} denote the closest zero of $p_n(x)$ to x. We define x_{jn} to be the closest zero of $p_n(x)$ on the left, in the event that x lies midway between two zeros.

The fundamental polynomials of Lagrange interpolation are

$$l_{kn}(x) = \lambda_{kn} \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(x_{kn})}{x - x_{kn}}, \qquad k = 1, 2, ..., n, \tag{2.3}$$

and the Lagrange interpolation polynomial of degree at most $n-1$ is

$$L_n(f; x) = \sum_{k=1}^{n} l_{kn}(x) f(x_{kn}). \tag{2.4}$$

For convenience we define

$$H_{n,p}(x)=\sum_{k=1}^{n}\lambda_{kn}|K_n(x,x_{kn})|^p \qquad \text{for } p>0,\ n=1,2,\ldots.$$

Let $f(x)$ be a bounded measurable function on $(-\infty,\infty)$. We define the rth modulus of continuity of f by

$$\omega_r(f;\delta)=\sup_{\substack{|h|\leqslant\delta\\ -\infty<x<\infty}}\left|\sum_{v=0}^{r}\binom{r}{v}(-1)^r f(x+vh)\right|, \qquad \delta>0.$$

We use the usual norm notation. For example,

$$\|f\|_\infty=\sup_{x\in\mathbb{R}}|f(x)|.$$

Throughout $c, c_1, c_2, \ldots$ will denote positive constants independent of n and x. For notational convenience the constants will not be numbered except in a case where confusion may arise. Thus c does not necessarily denote the same constant from line to line.

By $f(x)\sim g(x)$ we denote the condition $c_1\leqslant f(x)/g(x)\leqslant c_2$ for all relevant x.

The usual o, O notation will be used.

3. Main Results

The class of weights considered is as follows:

Definition 3.1. $W^2=\exp(-2Q)$ is a regular weight if it satisfies

(a) *Explicit Assumptions.* Q is an even, convex twice differentiable function in $(-\infty,\infty)$ with $Q(x)>0$ and $Q'(x)>0$ for $x\in(0,\infty)$ and

$$xQ''(x)/Q'(x)\leqslant c, \qquad 0<x<\infty, \tag{3.1}$$

$$0\leqslant Q''(x_1)\leqslant(1+c_1)\,Q''(x_2), \qquad 0<x_1<x_2, \tag{3.2}$$

$$Q'(2x)/Q'(x)>1+c, \qquad x \text{ large enough}. \tag{3.3}$$

(b) *Implicit Assumptions.*

$$|p_n(W^2,x)\,W(x)|\leqslant c_1q_n^{-1/2}, \qquad |x|\leqslant c_2q_n, \quad n\geqslant 1. \tag{3.4}$$

The explicit assumptions can be weakened substantially for the required properties of W^2 to hold. In fact (3.3) is implied by the other conditions on Q. However, for ease of reference, we retain the restrictions in the above form.

The implicit assumption (3.4) is essential for our own proofs. We note in particular, that the weights

$$W_m(x) = \exp(-\tfrac{1}{2} x^m), \qquad m = 2, 4, 6, ..., \tag{3.5}$$

satisfy (3.4) (Bonan [3], Nevai [24]). Of course for $W_m(x)$, we have by (2.1)

$$q_n = (2n/m)^{1/m}. \tag{3.6}$$

In addition it has been proved [1, Theorem 3.5; 10, Chap. 2] that if W^2 satisfies the explicit assumptions of Definition 3.1

$$a_n = \gamma_{n-1}/\gamma_n \sim q_n. \tag{3.7}$$

We shall prove as a consequence of (3.1) to (3.4) that W^2 satisfies

$$|p_{n-1}(x_{kn})\, W(x_{kn})| \sim q_n^{-1/2}, \qquad |x_{kn}| < cq_n. \tag{3.8}$$

The results on Lagrange interpolation can now be stated.

THEOREM 3.2. *Let W^2 be a regular weight. For all uniformly continuous functions $f(x)$ and all natural r, there exists c_1 such that uniformly for $|x| < c_1 q_n$,*

$$|f(x) - L_n(f; x)| \leqslant c_r\, \omega_r(f; q_n/n)\{(n/q_n)|x - x_{jn}|[\log n + W^{-1}(x)] + c\}. \tag{3.9}$$

THEOREM 3.3. *Let $W(x) = W_m(x)$, $m = 2, 4, 6, ...$ Let $\varepsilon > 0$. Let r be a positive integer. For all uniformly continuous functions $f(x)$ there exist c_2 and $c_1 > 1$ with the following properties:*

$$|f(x) - L_n(f; x)| \leqslant \begin{cases} c_r[\log n + W^{-1}(x)]\, \omega_r(f; q_n/n), & |x| \leqslant c_2 q_n, \\ \quad x \notin (2a_n - \varepsilon n^{-1/m},\ \ 2a_n + \varepsilon n^{-1/m}) & \\ \|f\|\; W^{-1}(x)\, c_1^{-n}, & |x| > c_2 q_n. \end{cases} \tag{3.10}$$

In the case of the Hermite weight we need not omit the interval $(2a_n - \varepsilon n^{-1/2}, 2a_n + \varepsilon n^{-1/2})$. For $m > 2$, the results of Bonan and Clark [4] can be used to fill the gap.

The following result shows that we can define a function $f(x)$ for which the rates of convergence of Theorem 3.2 are substantially best possible for all $|x| < cq_n$.

THEOREM 3.4. *Let W^2 be a regular weight. Then there exists c_2 and functions f_n depending on n and x, $n = 1, 2, \ldots$, such that for $|x| < c_1 q_n$,*

$$|f_n(x) - L_n(f; x)| \geqslant c 2^{-r} \omega_r(f_n; q_n/n)\{(n/q_n)|x - x_{jn}|[\log n + (1+|x|)^{-1} W^{-1}(x)] - 1\}. \tag{3.11}$$

In order to prove these results we need to obtain upper and lower bounds for the Lebesgue function

$$\sum_{k=1}^{n} |l_{kn}(x)|. \tag{3.12}$$

In fact we will obtain bounds for the generalized Lebesgue function

$$H_{n,p}(x) \equiv \sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{kn})|^p, \tag{3.13}$$

for $0 < p \leqslant 2$. $H_{n,1}(x)$ is the Lebesgue function (3.12). For the Hermite weight, $|x| < cn^{1/2}$, the upper bound (3.15) is better than that of [6, Satz 1] for $|x - x_{jn}| = o(q_n/n)$. Also, the lower bound of Freud [6, Satz 3] for $H_{n,1}(\xi)$, holds only for ξ a zero of $p_{n+1}(x)$, for which $2 \leqslant |\xi| \leqslant c_1 n^{1/2}$. In Theorem 3.6 we obtain a lower bound for $H_{n,p}(x)$, $0 < p \leqslant 2$, which holds for all $|x| < c_1 n^{1/2}$.

These results are stated as follows:

THEOREM 3.5. *Let $0 < p \leqslant 2$. Let W^2 be a regular weight. Then there exists c_1 such that uniformly for $|x| \leqslant c_1 q_n$,*

$$\text{(i)} \quad H_{n,p}(x) \leqslant c\{(n/q_n)^{p-1} W^{2-2p}(x) + [(n/q_n)|x - x_{jn}|]^p W^{-p}(x)\}, \qquad p \neq 1. \tag{3.14}$$

$$\text{(ii)} \quad H_{n,1}(x) \leqslant c(n/q_n)|x - x_{jn}|\{\log n + W^{-1}(x)\} + c_2. \tag{3.15}$$

The following results show that the upper bounds in Theorem 3.5 are substantially best possible.

THEOREM 3.6. *Let $0 < p \leqslant 2$. Let W^2 be a regular weight. Then there exists c_1 such that uniformly for $|x| \leqslant c_1 q_n$,*

$$\text{(i)} \quad H_{n,p}(x) \geqslant c\{n/q_n)^{p-1} W^{2-2p}(x) + [(n/q_n)|x - x_{jn}|]^p (1+|x|)^{-p} W^{-p}(x)\}, \qquad p \neq 1.$$

$$\text{(ii)} \quad H_{n,1}(x) \geqslant c(n/q_n)|x - x_{jn}|\{\log n + (1+|x|)^{-1} W^{-1}(x)\} + c_2. \tag{3.16}$$

Both of these theorems rely on the following $\sim$ relationship for the orthonormal polynomial:

THEOREM 3.7. *Let W^2 be a regular weight. Then there exists c such that uniformly for $|x| < cq_n$,*

$$|p_n(x)\, W(x)| \sim |x - x_{jn}|(n/q_n)\, q_n^{-1/2}. \tag{3.17}$$

The proofs of Theorems 3.5, 3.6 and 3.7 appear in Section 4. Theorems 3.2, 3.3 and 3.4 on Lagrange interpolation are proved in Section 5.

4. BOUNDS FOR $H_{n,p}(x)$

We need a number of preliminary results.

LEMMA 4.1. *If $W^2(x)$ satisfies the explicit assumptions of Definition 3.1 then the following results hold:*

$$\text{(a) } K_n(x, x) \leqslant c(n/q_n)\, W^{-2}(x), \qquad x \in \mathbb{R}. \tag{4.1}$$

(b) *There exists c_2 such that*

$$K_n(x, x) \geqslant c(n/q_n)\, W^{-2}(x), \qquad |x| \leqslant c_2 q_n. \tag{4.2}$$

$$\text{(c) } x_{1n} \leqslant cq_n. \tag{4.3}$$

(d) *There exists c_2 such that for $x_{k-1,n}, x_{kn} \in [-c_2 q_n, c_2 q_n]$,*

$$c_1 q_n/n < x_{k-1,n} - x_{k,n} < c_3 q_n/n. \tag{4.4}$$

Proof. (a) This is Lemma 2.5 in [8, p. 25].
(b) This is Lemma 4.2 in [8, p. 33].
(c) This follows from Theorem 1 in [7, p. 49].
(d) This follows from Theorem 5.1 in [8, p. 36]. ∎

LEMMA 4.2. *Let W^2 satisfy the explicit assumptions of Definition 3.1. Then there exists c such that for $x \in [-cq_n, cq_n]$,*

$$\text{(a)} \quad W(x_{jn}) \sim W(x) \sim W(x_{j+1,n}). \tag{4.5}$$

$$\text{(b)} \quad \lambda_{jn} \sim \lambda_n(x) \sim \lambda_{j+1,n}. \tag{4.6}$$

Proof. See Lemma 4.2 in Knopfmacher [11]. ∎

LEMMA 4.3 (Properties of Freud weights).

$$\text{(i)}\quad c_1 x^2 \leqslant Q(x) \leqslant c_2 x^{1+c}, \qquad x \geqslant c_3. \tag{4.7}$$

$$\text{(ii)}\quad c_1 x^{1/(1+c)} \leqslant q_x \leqslant c_2 x^{1/2}, \qquad x \geqslant c_3. \tag{4.8}$$

(iii) *If* $w > 1$ *then uniformly for* $1 \leqslant v < w$;

$$Q'(vx) \sim Q'(x), \qquad x \geqslant c_3. \tag{4.9}$$

$$\text{(iv)}\quad Q(x) \sim xQ'(x) \sim x^2 Q''(x), \qquad x \geqslant c_3. \tag{4.10}$$

$$\text{(v)}\quad c_1 x \leqslant Q'(x) \leqslant c_2 x^c, \qquad x \geqslant c_3. \tag{4.11}$$

Proof. (i) As Q'' is nondecreasing and positive, it is easily seen that $Q(x) \geqslant c_1 x^2$ for large x. The upper bound follows from Lemma 7(v) in [15].

(ii) The upper bound is Lemma 4.2g in [11]. For the lower bound see Lemma 7(viii) in [15].

(iii) This is Lemma 7(ix) in [15].

(iv) The first part of (4.10) is Lemma 7(vi) in [15]. It suffices to establish

$$Q'(x) \sim xQ''(x), \qquad x \geqslant c_3.$$

In view of (3.1), it suffices to shown

$$Q'(x) \leqslant cxQ''(x), \qquad x \geqslant c_3.$$

But for $x \geqslant 1$, by (3.2),

$$\begin{aligned} Q'(x) &= Q'(1) + \int_1^x Q''(u)\,du \\ &\leqslant Q'(1) + xQ''(x)(1 + c_1). \end{aligned}$$

Since $\lim_{x \to \infty} Q'(x) = \infty$, (by (4.7) and the first part of (4.10)) we obtain

$$(1+c)^{-1} \leqslant \liminf_{x \to \infty} xQ''(x)/Q'(x).$$

Hence, this completes the proof of (4.10).

(v) This now follows from (4.7) and (4.10). ∎

LEMMA 4.4. (Local Markov–Bernstein inequality). *Let* W^2 *satisfy the* *explicit assumptions in Definition* 3.1. *There exist c_1, c_2 and c_3 such that for $0<\delta\leqslant c_1$ and all polynomials P of degree at most n,*

$$\|P'W\|_{L_\infty(-\delta q_n,\,\delta q_n)}\leqslant c_3(n/q_n)\|PW\|_{L_\infty(-\delta c_2 q_n,\,\delta c_2 q_n)}. \tag{4.12}$$

Proof. Let ξ_n denote the positive root of the equation

$$\xi_n^2\max\{|Q''(u)|: 1\leqslant u\leqslant \xi_n\}=n, \tag{4.13}$$

n large enough. It is shown in Lubinsky [17, Corollary 3.2], that for all polynomials P of degree at most n, and all $0<\delta<\varepsilon\leqslant 1$,

$$\|P'W\|_{L_\infty(-\delta\xi_n,\,\delta\xi_n)}\leqslant c(n/\xi_n)\|PW\|_{L_\infty(-\varepsilon\xi_n,\,\varepsilon\xi_n)}. \tag{4.14}$$

It is easy to see that (4.14) implies (4.12), provided we can show

$$\xi_n\sim q_n,\qquad n \text{ large enough.} \tag{4.15}$$

Now by (3.2) and (4.13) above

$$\begin{aligned} q_nQ'(q_n)=n&\leqslant \xi_n^2Q''(\xi_n)(1+c_1)\\ &\sim \xi_n\,Q'(\xi_n)(1+c_1), \end{aligned}$$

by (4.10). Since Q' is nondecreasing, we obtain for some $c>1$,

$$q_n\,Q'(q_n)\leqslant(c\,\xi_n)\,Q'(c\,\xi_n)$$

so that

$$q_n\leqslant c\,\xi_n,\qquad n \text{ large enough.}$$

Next, from (4.13), we see

$$\begin{aligned} q_nQ'(q_n)=n&\geqslant \xi_n^2Q''(\xi_n)\\ &\geqslant c_2\,\xi_nQ'(\xi_n) \end{aligned}$$

(by (4.10), for some $c_2\leqslant 1$)

$$\geqslant(c_2\,\xi_n)\,Q'(c_2\,\xi_n),$$

as $c_2\leqslant 1$. We deduce $q_n\geqslant c_2\,\xi_n$. ∎

At this stage we can prove assertion (3.8).

LEMMA 4.5. *Assume W^2 is a regular weight. Then there exists c_1 such that*

$$|(p_{n-1}W)(x_{kn})|\sim q_n^{-1/2},\qquad |x_{kn}|\leqslant c_1\,q_n. \tag{4.16}$$

Proof. In view of (3.4) it suffices to show

$$|(p_{n-1}W)(x_{kn})| \geqslant cq_n^{-1/2}, \qquad |x_{kn}| \leqslant c_1 q_n. \tag{4.17}$$

We shall use the identity

$$K_n(x, x) = \gamma_{n-1}/\gamma_n \,(p_n'(x)\, p_{n-1}(x) - p_n(x)\, p_{n-1}'(x)), \tag{4.18}$$

which follows easily from the Christoffel–Darboux formula. Using (4.2) and (3.7) with $x = x_{kn}$, we obtain

$$c_3(n/q_n)\, W^{-2}(x_{kn}) \leqslant c_4\, q_n\, p_n'(x_{kn})\, p_{n-1}(x_{kn}). \tag{4.19}$$

Now by the local Markov–Bernstein inequality, we have

$$\begin{aligned} \| p_n' W \|_{L_\infty(-\delta q_n, \delta q_n)} &\leqslant c(n/q_n) \| p_n W \|_{L_\infty(-\delta c_1 q_n, \delta c_1 q_n)} \\ &\leqslant cn/q_n^{3/2}, \end{aligned} \tag{4.20}$$

by (3.4), if δ is small enough. Then (4.19) and (4.20) yield

$$c_3\, n/q_n \leqslant c_5\, n/q_n^{1/2}\, |p_{n-1} W|(x_{kn}), \qquad |x_{kn}| \leqslant cq_n,$$

which yields (4.17). ∎

Proof of Theorem 3.7. Let $|x| < cq_n$. We use the technique of Nevai [22, p. 171].

$$\begin{aligned} l_{jn}^2(x) &= \lambda_{jn}^2\, K_n^2(x, x_{jn}) \\ &\leqslant \lambda_{jn}^2\, \lambda_{jn}^{-1}\, K_n(x, x) \end{aligned}$$

(by the Cauchy–Schwarz inequality)

$$\leqslant c,$$

by (4.6). Now either x lies between $x_{j-1,n}$ and x_{jn} or x lies between x_{jn} and $x_{j+1,n}$. Let for simplicity, $x_{jn} \leqslant x \leqslant x_{j-1,n}$. Then by Lemma 9.32 [22, p. 170],

$$l_{j-1,n}(x) + l_{jn}(x) \geqslant 1.$$

Now by (4.6) and (3.8),

$$|\lambda_{j-1,n}\, p_{n-1}(x_{j-1,n})| \sim |\lambda_{jn}\, p_{n-1}(x_{jn})|.$$

Also $l_{j-1,n}(x) \geqslant 0$, $l_{jn}(x) \geqslant 0$ and sign $p_{n-1}(x_{j-1,n}) = -\operatorname{sign} p_{n-1}(x_{jn})$. Hence using (2.3) and the fact that $|x - x_{j-1,n}| \geqslant |x - x_{jn}|$,

$$l_{j-1,n}(x) \leqslant c l_{jn}(x).$$

Therefore for $|x| \leqslant cq_n$, in all possible cases,

$$l_{jn}^2(x) \sim 1.$$

Hence by (2.3)

$$\lambda_{jn}^2 (\gamma_{n-1}/\gamma_n)^2 |p_n(x)/(x - x_{jn})|^2 \, p_{n-1}^2(x_{jn}) \sim 1. \tag{4.21}$$

The result now follows by (3.8), (3.7), (4.1) and (4.2). ∎

LEMMA 4.6. *Let* $0 < p < 2$. *Let* W^2 *satisfy the explicit assumptions in Definition* 3.1. *Then*

$$\sum_{k=1}^{n} \lambda_{kn} W^{-p}(x_{kn}) \leqslant c, \qquad n = 1, 2, ... \tag{4.22}$$

Proof. It is obviously sufficient to show that

$$\lim_{n \to \infty} \sum_{k=1}^{n} \lambda_{kn} W^{-p}(x_{kn}) = \int_{-\infty}^{\infty} W^{-p}(x) \, W^2(x) \, dx. \tag{4.23}$$

To see this we note that Theorem 4 in [15] shows that there exists an entire function $G(x)$ with the following properties:

$$G(x) \sim W^{-p}(x) \, x^2 \qquad \text{as } x \to \infty;$$

$$G^{(2n)}(x) \geqslant 0, \qquad n = 1, 2, ..., x \in \mathbb{R}$$

and

$$\int_{-\infty}^{\infty} G(x) \, W^2(x) \, dx < \infty.$$

Since

$$\lim_{n \to \infty} W^{-p}(x)/G(x) = 0,$$

we can apply Theorem III.1.6 [5, p. 93] to prove (4.22). ∎

Without further mention let W^2 denote a regular weight in what follows.

LEMMA 4.7. *There exists* c_1 *such that*

(i) *for* $|x|, |x_{kn}| \leqslant c_1 q_n$,

$$|K_n(x, x_{kn})| \sim (n/q_n) |x - x_{jn}| \, W^{-1}(x) \, W^{-1}(x_{kn})/|x - x_{kn}|, \tag{4.24}$$

(ii) *for* $|x| < c_1 q_n$, $|x_{kn}| > c_2 q_n$, $c_2 > c_1$, *we have*

$$|K_n(x, x_{kn})| \sim q_n^{-1/2}(n/q_n)|x - x_{jn}| \, W^{-1}(x)|p_{n-1}(x_{kn})|. \tag{4.25}$$

Proof. (i) It follows from (2.2), (3.7), (3.8) and (3.17) that

$$\begin{aligned}|K_n(x, x_{kn})| &= \left|\frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x)\, p_{n-1}(x_{kn})}{x - x_{kn}}\right| \\ &\sim q_n(q_n^{-1/2} W^{-1}(x_{kn}))((n/q_n)|x - x_{jn}| \, q_n^{-1/2} W^{-1}(x))/|x - x_{kn}| \\ &\sim (n/q_n)|x - x_{jn}| \, W^{-1}(x) \, W^{-1}(x_{kn})/|x - x_{kn}|.\end{aligned}$$

(ii) This follows similarly from (2.2), (3.7), (3.17), (4.3) and since

$$|x - x_{kn}| > (c_2 - c_1)\, q_n = c q_n. \quad ∎$$

LEMMA 4.8. *Let* $0 < p \leqslant 2$. *Let*

$$H_{n,p}^{(1)}(x) = \sum_{\substack{|x - x_{kn}| \leqslant 1 \\ k \neq j}} \lambda_{kn} |K_n(x, x_{kn})|^p, \qquad x \in \mathbb{R}. \tag{4.26}$$

Then there exists c_1 *such that for* $|x| \leqslant c_1 q_n$,

(a) $H_{n,p}^{(1)}(x) \leqslant c[(n/q_n)|x - x_{jn}|]^p \{(n/q_n)^{p-1} W^{2-2p}(x) + W^{-p}(x)\},$

$$1 < p \leqslant 2, \tag{4.27}$$

(b) $$H_{n,1}^{(1)}(x) \leqslant c(n/q_n)|x - x_{jn}| \{\log n + W^{-1}(x)\}, \tag{4.28}$$

(c) $$H_{n,p}^{(1)}(x) \leqslant c[(n/q_n)|x - x_{jn}|]^p, \qquad 0 < p < 1. \tag{4.29}$$

Proof. First note that for $p = 2$, by the Gauss-quadrature formula

$$\begin{aligned}H_{n,p}(x) &= \sum_{k=1}^{n} \lambda_{kn} K_n^2(x, x_{kn}) \\ &= \int_{-\infty}^{\infty} K_n^2(x, t)\, W^2(t)\, dt \\ &= K_n(x, x) \leqslant c(n/q_n)\, W^{-2}(x), \qquad |x| \leqslant c_1 q_n,\end{aligned} \tag{4.30}$$

by (4.1). So we may assume $p < 2$. Next, we note that it suffices to consider $x \geqslant 0$, as $H_{n,p}^{(1)}(x)$ is an even function. To see this we note that

$$K_n(-x, -x_{kn}) = K_n(x, x_{kn}),$$

since each orthogonal polynomial is either even or odd and further the zeros $x_{nn}, ..., x_{1n}$ are symmetric about 0. So let $x \geqslant 0$, and let $u \in (|x - x_{kn}|, |x - x_{kn}| + q_n/n)$. If first $x > x_{kn}$, then

$$\begin{aligned} x - u &\in (x - |x - x_{kn}| - q_n/n, \quad x - |x - x_{kn}|) \\ &= (x_{kn} - q_n/n, x_{kn}). \end{aligned}$$

Since $W(y) \sim W(y + q_n/n)$ for $|y| \leqslant c_1 q_n$, we have

$$W(x - u) \sim W(x_{kn}). \tag{4.31}$$

Secondly, if $x < x_{kn}$, then

$$\begin{aligned} x + u &\in (x + |x - x_{kn}|, \quad x + |x - x_{kn}| + q_n/n) \\ &= (x_{kn}, x_{kn} + q_n/n). \end{aligned}$$

Hence

$$W(x + u) \sim W(x_{kn}). \tag{4.32}$$

Now as $x \geqslant 0$ and $u > 0$, $|x - u| \leqslant x + u$, so that $W(x + u) \leqslant W(x - u)$. Then by (4.31) and (4.32),

$$W(x_{kn}) \leqslant c W(x - u), \tag{4.33}$$

for $u \in (|x - x_{kn}|, |x - x_{kn}| + q_n/n)$ and uniformly for $0 \leqslant x < c_1 q_n$. Furthermore for such u, if $k \neq j$, (4.4) shows that

$$u \leqslant q_n/n + |x - x_{kn}| \leqslant c|x - x_{kn}|. \tag{4.34}$$

Next by (4.2), (4.24) and (4.26),

$$\begin{aligned} H^{(1)}_{n,p}(x) &\leqslant c(q_n/n)[(n/q_n)|x - x_{jn}|]^p W^{-p}(x) \sum_{\substack{|x - x_{kn}| \leqslant 1 \\ k \neq j}} W^{2-p}(x_{kn})/|x - x_{kn}|^p \\ &\leqslant c[(n/q_n)|x - x_{jn}|]^p W^{-p}(x) \\ &\quad \times \sum_{\substack{|x - x_{kn}| \leqslant 1 \\ k \neq j}} \int_{|x - x_{kn}|}^{|x - x_{kn}| + q_n/n} W^{2-p}(x - u) u^{-p} \, du \end{aligned}$$

(by (4.33) and (4.34))

$$\leqslant c[(n/q_n)|x - x_{jn}|]^p W^{-p}(x) \int_{q_n/n}^{1 + q_n/n} W^{2-p}(x - u) u^{-p} \, du. \tag{4.35}$$

Here we have used the fact that each interval of the form $J_k = (|x - x_{kn}|, |x - x_{kn}| + q_n/n)$ can intersect at most two intervals of the form $I_l = (lq_n/n, (l+1)\, q_n/n)$, $l = 0, 1, 2, \ldots$, and is contained in the union of at most two such intervals. Furthermore, it follows from (4.4) that the number of intervals J_k intersecting any I_l is bounded above independent of x, at least for $|x| < c_1 q_n$ and some suitable c_1. Next by (4.35)

$$H_{n,p}^{(1)}(x) \leqslant c[(n/q_n)|x - x_{jn}|]^p\, W^{2-2p}(x) \int_{q_n/n}^{2} W^{p-2}(x)\, W^{2-p}(x-u)\, u^{-p}\, du. \tag{4.36}$$

If firstly $x \leqslant 2$, then $W^{p-2}(x)\, W^{2-p}(x-u) \leqslant c$ for all $u \in [q_n/n, 2]$ and so

$$H_{n,p}^{(1)}(x) \leqslant c[(n/q_n)|x - x_{jn}|]^p\, W^{2-2p}(x) \int_{q_n/n}^{2} u^{-p}\, du. \tag{4.37}$$

For $x \leqslant 2$ (4.27), (4.28) and (4.29) follow easily from (4.37) as $W^{2-2p}(x) \leqslant 1$, if $0 < p \leqslant 1$. Note too, that $\log(n/q_n) \sim \log n$, by (4.8). Let us suppose now $x > 2$. Then for $u \in [q_n/n, 2]$,

$$\begin{aligned} W^{p-2}(x)\, W^{2-p}(x-u) &= \exp\left((2-p) \int_{x-u}^{x} Q'(t)\, dt \right) \\ &\leqslant \exp c\, u\, Q'(x) \end{aligned} \tag{4.38}$$

as Q' is positive and nondecreasing and $x > x - u > x - 2 > 0$. Then by (4.36) and (4.38) for $2 \leqslant x \leqslant c_1 q_n$,

$$\begin{aligned} H_{n,p}^{(1)}(x) &\leqslant c[(n/q_n)|x - x_{jn}|]^p\, W^{2-2p}(x) \int_{q_n/n}^{2} \exp(c\, u\, Q'(x))\, u^{-p}\, du \\ &= c[(n/q_n)|x - x_{jn}|]^p\, W^{2-2p}(x) \int_{cQ'(x)\, q_n/n}^{2cQ'(x)} e^{w}\, w^{-p}\, dw (cQ'(x))^{p-1} \end{aligned}$$

(by the substitution $w = cQ'(x)\, u$)

$$\begin{aligned} &\leqslant c[n/q_n\, |x - x_{jn}|]^p\, W^{2-2p}(x) (Q'(x))^{p-1} \\ &\quad \times \left[\int_{cQ'(x)\, q_n/n}^{1} w^{-p}\, dw + \int_{1}^{2cQ'(x)} e^{w}\, dw \right]. \end{aligned} \tag{4.39}$$

In the case where $cQ'(x)\, q_n/n > 1$ or $2cQ'(x) < 1$, we omit, respectively, the first and second integrals in (4.39). Now let $1 < p < 2$. Then from (4.39) for $2 \leqslant x \leqslant c_1 q_n$,

$$H_{n,p}^{(1)}(x) \leqslant c[(n/q_n)|x - x_{jn}|]^p\, W^{2-2p}(x) [(n/q_n)^{p-1} + (Q'(x))^{p-1}\, e^{2cQ'(x)}]. \tag{4.40}$$

From (4.27) and (4.40), we see that it suffices to prove that

$$W^{2-p}(x)\,(Q'(x))^{p-1}\,e^{2cQ'(x)} < c, \qquad x \in [0, \infty) \tag{4.41}$$

in order to complete the proof of (4.27). But the left member of (4.41) equals

$$\begin{aligned}&\exp(-(2-p)\,Q(x) + (p-1)\log Q'(x) + 2cQ'(x))\\ &\quad \leqslant \exp(Q(x)\{p-2+c/x\})\end{aligned}$$

(by (4.10) and (4.11))

$$\leqslant c_1,$$

since for large x, $p-2+c/x<0$. Hence we have proved (4.41) and also (4.27), for $|x| \leqslant c_1 q_n$. The cases $p=1$ and $0<p<1$, follow similarly from (4.39). If $p<1$, we use the fact that $W^{2-2p}(w)<1$. ∎

LEMMA 4.9. *Let $0<p<2$. There exists c such that for $|x|<cq_n$,*

$$\lambda_{jn}\,|K_n(x, x_{jn})|^p \sim (n/q_n)^{p-1}\,W^{2-2p}(x). \tag{4.42}$$

Proof. We have by (4.1), (4.2) and (4.24) with $k=j$,

$$\begin{aligned}\lambda_{kn}\,|K_n(x, x_{jn})|^p &\sim q_n/n\;W^2(x_{jn})[(n/q_n)\,W^{-1}(x)\,W^{-1}(x_{jn})]^p\\ &\sim (n/q_n)^{p-1}\,W^{2-2p}(x),\end{aligned}$$

by (4.5). ∎

LEMMA 4.10. *Let $0<p<2$. We define for a suitable choice of c,*

$$H^{(2)}_{n,p}(x) = \sum_{\substack{|x-x_{kn}|\geqslant 1\\ |x_{kn}|<cq_n}} \lambda_{kn}\,|K_n(x, x_{kn})|^p.$$

Then there exists c_1 such that uniformly for $|x|<c_1 q_n$,

$$H^{(2)}_{n,p}(x) \leqslant c[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x).$$

Proof. It follows from (4.24) and (4.22) that

$$\begin{aligned}H^{(2)}_{n,p}(x) &\leqslant c[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x)\sum_{k=1}^{n}\lambda_{kn}\,W^{-p}(x_{kn})\\ &\leqslant c[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x). \quad ∎\end{aligned}$$

LEMMA 4.11. *Let* $0<p<2$. *Let* c *be as in Lemma* 4.10 *and let*

$$H^{(3)}_{n,p}(x)=\sum_{|x_{kn}|>cq_n}\lambda_{kn}\,|K_n(x,x_{kn})|^p.$$

Then there exists $c_1<c$ *such that uniformly for* $|x|<c_1q_n$,

$$H^{(3)}_{n,p}(x)\leqslant cq_n^{-p/2}[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x).$$

Proof. By (4.25)

$$H^{(3)}_{n,p}(x)\leqslant c\sum_{|x_{kn}|>cq_n}\lambda_{kn}|p_{n-1}(x_{kn})|^p\,\{q_n^{-p/2}[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x)\}$$

$$\leqslant cq_n^{-p/2}\,[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x)\left(\sum_{|x_{kn}|>cq_n}\lambda_{kn}\,p^2_{n-1}(x_{kn})\right)^{p/2}$$

(by Hölder's inequality)

$$\leqslant c\,q_n^{-p/2}[(n/q_n)|x-x_{jn}|]^p\,W^{-p}(x).\quad ▮$$

Proof of Theorem 3.5. First let $0<p<2$. The upper bound now follows directly from Lemmas 4.8–4.11 for $|x|<c_1q_n$, n sufficiently large, and noting that by (4.4),

$$[(n/q_n)|x-x_{jn}|]^p\leqslant c. \tag{4.43}$$

The case $p=2$ follows from (4.30). ▮

Proof of Theorem 3.6. It suffices to consider $x\geqslant 0$ as $H_{n,p}$ is even. We suppose $x>2$. The proof for $0\leqslant x<2$ is similar. Now by (4.1) and (4.24) for x, $|x_{kn}|<c_1q_n$,

$$\begin{aligned}\lambda_{kn}|K_n(x,x_{kn})|^p&\geqslant c(q_n/n)\,W^2(x_{kn})\\&\quad\times[(n/q_n)|x-x_{jn}|\,W^{-1}(x)\,W^{-1}(x_{kn})/|x-x_{kn}|]^p\\&\geqslant cq_n/n\,W^{-p}(x)\,W^{2-p}(x_{kn})|x-x_{kn}|^{-p}[n/q_n\,|x-x_{jn}|]^p.\end{aligned} \tag{4.44}$$

Now let us consider the sum over all abscissas x_{kn} which fall in (0, 1). By the separation property of the zeros the number of such x_{kn} is order n/q_n. Therefore if $2<x<cq_n$,

$$\begin{aligned}\sum_{0<x_{kn}<1}\lambda_{kn}\,|K_n(x,x_{kn})|^p&\geqslant cW^{2-p}(1)\,W^{-p}(x)\,x^{-p}[n/q_n\,|x-x_{jn}|]^p\\&=cx^{-p}\,W^{-p}(x)[n/q_n\,|x-x_{jn}|]^p.\end{aligned} \tag{4.45}$$

Again by (4.44) and using the fact that by (4.4)

$$x_{kn} - x_{k+r,n} > crq_n/n,$$

$$\sum_{x-1 < x_{kn} < x} \lambda_{kn} \, |K_n(x, x_{kn})|^p$$

$$\geqslant cW^{2-2p}(x)\, q_n/n[n/q_n \, |x - x_{jn}|]^p \sum_{x-1 < x_{kn} < x} |x - x_{kn}|^{-p}$$

$$\geqslant cW^{2-2p}(x)\, q_n/n[n/q_n \, |x - x_{jn}|]^p \sum_{1 \leqslant r < c_2 n/q_n} |r\, q_n/n|^{-p}$$

$$\geqslant c(n/q_n)^{p-1}\, W^{2-2p}(x)[n/q_n \, |x - x_{jn}|]^p \sum_{1 \leqslant r < c_2 n/q_n} r^{-p}. \qquad (4.46)$$

Now it is easily seen that

$$\sum_{1 \leqslant r \leqslant c_2 n/q_n} r^{-p} \geqslant \begin{cases} (n/q_n)^{1-p}, & 0 < p < 1, \\ c \log n, & p = 1, \\ c, & 1 < p \leqslant 2. \end{cases} \qquad (4.47)$$

Now since $x > 2$, $(0, 1)$ and $(x-1, x)$ are disjoint intervals. The result for $p \neq 1$, now follows from (4.45) and (4.42). For $p = 1$ the result follows from (4.45), (4.42), (4.46), and (4.47). ∎

In order to prove Theorem 3.3 on Lagrange interpolation for the weights $W_m(x)$, we must derive as well, an upper bound for $H_{n,p}(x)$ for $|x| > c_1 q_n$. To this end we prove:

LEMMA 4.12. *For all* $x \in \mathbb{R}$,

(i) $$\sum_{|x_{kn}| > Q^{-1}(\log(n/q_n))} \lambda_{kn} \, |K_n(x, x_{kn})|^p \leqslant c(q_n/n)^{1-p}\, W^{-p}(x), \qquad 0 < p \leqslant 1.$$

(ii) *Let* c_1 *be an arbitrary constant. Then*

$$\sum_{|x_{kn}| > c_1 q_n} \lambda_{kn} \, |K_n(x, x_{kn})|^p \leqslant cW^{-p}(x), \qquad 1 < p < 2.$$

Proof. (i) Let $0 < p < 2$. It follows from Hölder's inequality, (4.30) and (4.22) that

$$\sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{kn})|^p W^{(p-2)/2}(x_{kn})$$
$$\leqslant \left(\sum_{k=1}^{n} \lambda_{kn} K_n^2(x, x_{kn})\right)^{p/2} \left(\sum_{k=1}^{n} \lambda_{kn} W^{((p-2)/2)(2/(2-p))}\right)^{(2-p)/2}$$
$$\leqslant K_n(x, x)^{p/2} \left(\sum_{k=1}^{n} \lambda_{kn} W^{-1}(x_{kn})\right)^{(2-p)/2}$$
$$\leqslant cK_n(x, x)^{p/2}. \tag{4.48}$$

Next assume $0<p\leqslant 1$. Now $|x_{kn}|>Q^{-1}(\log n/q_n)$ implies that $Q(x_{kn})\geqslant \log n/q_n$. Hence

$$W(x_{kn}) = \exp(-Q(x_{kn})) \leqslant q_n/n. \tag{4.49}$$

Therefore if $S=\{x_{kn}: |x_{kn}|\geqslant Q^{-1}(\log n/q_n)\}$

$$\sum_S \lambda_{kn} |K_n(x, x_{kn})|^p \leqslant \max_S \{W^{(2-p)/2}(x_{kn})\} \sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{kn})|^p W^{(p-2)/2}(x_{kn})$$
$$\leqslant c(q_n/n)^{(2-p)/2} K_n(x, x)^{p/2}$$

(by (4.48) and (4.49))

$$\leqslant c(q_n/n)^{(2-p)/2}((n/q_n) W^{-2}(x))^{p/2}$$

(by (4.1))

$$\leqslant c(n/q_n)^{p-1} W^{-p}(x).$$

(ii) Let $1<p<2$. Now $|x_{kn}|>c_1 q_n$ implies

$$Q(x_{kn}) \geqslant Q(c_1 q_n) \geqslant c_2 q_n^2,$$

by (4.7). Therefore by (4.8)

$$W(x_{kn}) = \exp(-Q(x_{kn})) \leqslant \exp(-c_3 n^{2/(1+c)}). \tag{4.50}$$

Hence for n sufficiently large,

$$\sum_{|x_{kn}|>c_1 q_n} \lambda_{kn} |K_n(x, x_{kn})|^p \leqslant \max_{|x_{kn}|>c_1 q_n} \{W^{(2-p)/2}(x_{kn})\}$$
$$\times \sum_{k=1}^{n} \lambda_{kn} |K_n(x, x_{kn})|^p W^{(p-2)/2}(x_{kn})$$
$$\leqslant (n/q_n)^{p/2} \exp(-c((2-p)/2)\, n^{2/(1+c)})\, W^{-p}(x)$$

(by (4.48), (4.1) and (4.50))

$$\leqslant cW^{-p}(x). \quad \blacksquare$$

THEOREM 4.13. *Let* $W(x)=W_m(x)$, $m=2, 4, 6, \ldots$. *In the case* $W(x)=W_2(x)$ *let* $x\in\mathbb{R}$. *Otherwise let* $\varepsilon>0$ *and let* $x\in\mathbb{R}\setminus(2a_n-\varepsilon n^{-1/m}, 2a_n+\varepsilon n^{-1/m})$. *Then for* $n=1, 2, 3, \ldots$

$$\text{(i)}\quad H_{n,p}(x)\leqslant c\{(n^{1-1/m})^{p-1}W^{2-2p}(x)+W^{-p}(x)\}, \qquad 1<p\leqslant 2.$$

$$\text{(ii)}\quad H_{n,1}(x)\leqslant c\{\log n+W^{-1}(x)\}. \tag{4.51}$$

$$\text{(iii)}\quad H_{n,p}(x)\leqslant c\,W^{-p}(x), \qquad 0<p<1.$$

Proof. The results for $|x|<c_1 q_n$ follow from Theorem 3.5, (3.6) and (4.43). Therefore let $|x|\geqslant c_1 q_n$. Now by (4.7) for n sufficiently large,

$$Q'^{-1}(\log(n/q_n))<c(\log(n/q_n))^{1/2}<c(\log n)^{1/2}<c_2 q_n.$$

It follows from Lemma 4.12 for $0<p<2$ that we need only consider the sum over abscissas x_{kn} which satisfy $|x_{kn}|<c_2q_n$, $c_2<c_1$. Now by (2.2), (3.7), (3.8) and (4.22)

$$\begin{aligned}\sum_{|x_{kn}|<c_2q_n}\lambda_{kn}\,|K_n(x,x_{kn})|^p &\leqslant c\,q_n^{-p/2}\,|p_n(x)|^p\sum_{k=1}^{n}\lambda_{kn}W^{-p}(x_{kn})\\ &\leqslant c\,q_n^{-p/2}\,|p_n(x)|^p.\end{aligned} \tag{4.52}$$

At this point we require a bound for $|p_n(x)|$, $|x|>c_1q_n$. For the weights $W_m(x)$ Lubinsky [16] proved the following inequality:

$$W_m^2(x)\,p_n^2(x)|1-|x|^2/(2a_n)^2|\leqslant cn^{-1/m}, \qquad x\in\mathbb{R}.$$

We deduce from this that for $x\in\mathbb{R}\setminus(2a_n-\varepsilon n^{-1/m}, 2a_n+\varepsilon n^{-1/m})$

$$p_n^2(x)\,W_m^2(x)\leqslant cn^{1/m}, \tag{4.53}$$

where the constant in (4.53) depends on $\varepsilon>0$. If $m>2$ the result follows from (3.6), (4.52) and (4.53) for n sufficiently large. In the case $m=2$, (4.53) holds for all $x\in\mathbb{R}$, [26, p. 242, equation 8.91.10]. Hence the result by (3.6) and (4.52). To extend the results to all n, we note that by Hölder's inequality and (4.1) for $n<n_0$, n_0 fixed,

$$\begin{aligned}H_{n,p}(x)&\leqslant c\,K_n(x,x)^{p/2}\\ &\leqslant c\,n_o^{p/2}\,W^{-p}(x)\\ &=c\,W^{-p}(x). \quad \blacksquare\end{aligned}$$

For $W(x) = W_m(x)$, $m > 2$, the results of Bonan and Clark [4] may be used to fill the gap $(2a_n - \varepsilon n^{-1/m}, 2a_n + \varepsilon n^{-1/m})$.

5. Pointwise Convergence of Lagrange Interpolation

We now apply the bounds for the Lebesgue function $H_{n,1}(x)$, to prove pointwise convergence of Lagrange interpolation for uniformly continuous functions $f(x)$.

Proof of Theorem 3.2. Throughout the proof we use c_1 to denote a constant for which Theorem 3.5 is valid. Furthermore we use c_2 to denote a fixed constant which satisfies $c_2 > 2$ and $x_{1n} < c_2 q_n$. Now let $f_n(x) = f(c_2 q_n x)$. Then

$$\omega_r(f_n, \delta) = \omega_r(f, c_2 q_n \delta) \leqslant c\, \omega_r(f, q_n \delta). \tag{5.1}$$

Now we can find a polynomial $P(x)$ of degree $\leqslant n-1$ so that for $|x| \leqslant 1$, $|P_{n-1}(x)| \leqslant 2\|f_n\|_\infty$ and by Jackson's theorem (see Lorentz [13, p. 58, equation 10] for a proof for trigonometric polynomials)

$$\begin{aligned} |f_n(x) - P_{n-1}(x)| &\leqslant c_r\, \omega_r(f_n, n^{-1}) \\ &\leqslant c\, c_r\, \omega_r(f, q_n/n), \end{aligned}$$

by (5.1). Thus if $P^*_{n-1}(x) = P_{n-1}(x/c_2 q_n)$, $x \in \mathbb{R}$,

$$|f(x) - P^*_{n-1}(x)| \leqslant c\, c_r\, \omega_r(f; q_n/n), \qquad |x| < c_2 q_n. \tag{5.2}$$

Now by (5.2), (3.15) and the identity

$$P_{n-1}(x/c_2 q_n) = \sum_{k=1}^{n} l_{kn}(x)\, P_{n-1}(x/c_2 q_n),$$

it follows that for $|x| < c_1 q_n$,

$$\begin{aligned} &\Big|f(x) - \sum_{k=1}^{n} l_{kn}(x) f(x_{kn})\Big| \\ &\leqslant |f(x) - P_{n-1}(x/c_2 q_n)| + \sum_{k=1}^{n} |l_{kn}(x)|\, |f(x_{kn}) - P_{n-1}(x_{kn}/c_2 q_n)| \qquad (5.3) \\ &\leqslant c_r\, \omega_r(f; q_n/n)\{(n/q_n)|x - x_{jn}|[\log n + W^{-1}(x)] + c\}. \quad \blacksquare \end{aligned}$$

Proof of Theorem 3.3. Let c_2 be as in the proof of Theorem 3.2 above. As inequality (5.3) is valid for $|x| < c_2 q_n$ we can apply (4.51) to obtain the upper half of (3.10). By Theorem 4.13 we need not omit the interval of length $2\varepsilon n^{-1/m}$ around $2a_n$ in the case $m=2$. We prove the result for $|x|>c_2q_n$ as follows:

$$|f(x)-L_n(f;x)| \leqslant |f(x)|+|L_n(f;x)|$$
$$\leqslant W(c_2q_n)\|f\| W^{-1}(x)+|L_n(f;x)|. \tag{5.4}$$

As $c_2>1$,

$$W_m(c_2q_n)<W_m(q_n)=e^{-n/m}. \tag{5.5}$$

Also, by the infinite–finite range inequality [Lubinsky, 14, Theorem A],

$$|x^n L_n(f;x)\, W(x)| \leqslant \max_{x\in\mathbb{R}} |x^n L_n(f;x)\, W(x)|$$
$$\leqslant \max_{|x|\leqslant cq_n} |x^n L_n(f;x)\, W(x)|$$
$$\leqslant (cq_n)^n \max_{|x|\leqslant cq_n} |L_n(f;x)\, W(x)|.$$

Therefore,

$$|L_n(f;x)| \leqslant (cq_n/x)^n\, W^{-1}(x) \max_{|x|\leqslant cq_n} |L_n(f;x)\, W(x)|$$
$$\leqslant (cq_n/x)^n\, W^{-1}(x)\|f\| \max_{|x|\leqslant cq_n} \left\{\sum_{k=1}^{n} \lambda_{kn}\, |K_n(x,x_{kn})|\, W(x)\right\}$$
$$\leqslant c_3(cq_n/x)^n\, W^{-1}(x)\|f\| \max_{|x|\leqslant cq_n} \{K_n(x,x)\, W^2(x)\}^{1/2}$$

(by the Cauchy–Schwarz inequality)

$$\leqslant c_3(n/q_n)^{1/2}(cq_n/x)^n\|f\|\, W^{-1}(x)$$

(by (4.1))

$$\leqslant c_3\, c_1^{-n}\, \|f\|\, W^{-1}(x), \qquad |x|>c_2q_n, \tag{5.6}$$

c_2 sufficiently large. The lower half of (3.10) now follows from (5.4), (5.5) and (5.6). ▮

Proof of Theorem 3.4. We define the function $f_n(t)$ as follows. Let $f_n(x_{kn})=\operatorname{sign} l_{kn}(x)$ and let f_n be continuous between the zeros x_{kn}, $x_{k+1,n}$ and satisfy $\|f_n\|\leqslant 1$. For example, let f_n interpolate linearly between $x_{k+1,n}$ and x_{kn} and let $f_n(t)=f_n(x_{1n})$, $t>x_{1n}$ and $f_n(t)=f_n(x_{nn})$, $t<x_{nn}$.

Then

$$\omega_r(f_n; \delta) \leqslant 2^r \|f_n\| \leqslant 2^r \tag{5.7}$$

and

$$L_n(f_n; x) = \sum_{k=1}^{n} |l_{kn}(x)|. \tag{5.8}$$

Also

$$|L_n(f_n; x) - f(x)| \geqslant \|f_n\| [H_{n,1}(x) - 1].$$

The result now follows if n is sufficiently large, by applying (5.7) and (3.16) to the above. ∎

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